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New Proof and Extension of Sarkovskii's Theorem

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Elementary periodic orbits are introduced and are shown to obey a total order. The total ordering of the elementary periodic orbits provides the basis for the proof of Theorem (SE), the principal result of this report. Theorem (SE) is an extension of Sarkovskii's Theorem and subsumes it. Theorem (SE) incorporates the notion of $E(\infty)$ as an infinite preorbit which is directly related to the notion of turbulence. A paper with the same title was submitted for publication in the *Journal of Mathematical Analysis and Applications.*

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1. Introduction.

Examples of what we shall call an elementary periodic orbit abound in the considerable literature [e.g. 1,2,3,4] related to the proof of Sarkovskii's theorem. Almost all such examples are of odd period. No formal definition for all periods has been advanced. We define elementary periodic orbits, give a systematic study, and establish that they are totally ordered. The total ordering of the elementary periodic orbits provides the basis for the proof of Theorem (SE), an extension of Sarkovskii's theorem that subsumes Sarkovskii's theorem. The purpose of this note is to prove Theorem (SE). It is stated in Section 3. and proved in the final Section 8.. Theorem (SE) emphasizes the important role played by the elementary periodic orbits. The simultaneous existence of these special orbits of all periods is a rule rather than an exception. For example, if f has an elementary periodic orbit of period $n \neq 2, 4$, then f^2 has elementary periods of all periods. Furthermore, the existence of any elementary orbit of period $n \neq 2, 4$ ensures the existence of infinitely many distinct periodic orbits of f. This is in contrast to Sarkovskii's theorem which guarantees infinitely many periodic orbits only when $n \neq 2^m$, $m = 1, 2, \cdots$. In addition, Theorem (SE) incorporates the notion $E(\infty)$ as an infinite preorbit. This notion is directly related to the notion of turbulence introduced by Block and Coppel [5] since $E(\infty)$ implies that f^2 is necessarily turbulent.

2. Definitions and Notation.

Let $f : R \rightarrow R$ be continuous and $x_0 \in R$. The orbit of x_0 under f is defined as the set $\{x : x = f^n(x_0),\ n = 0, 1, \ldots\}$, where, for every positive integer n, f^n is the n-th iterate of f, $f^1 = f$, and $f^0(x_0) = x_0$. We shall write $x_n := f^n(x_0)$ for a given $x_0 \in R$ and call $x_1, x_2, \ldots$ the successors of x_0. A preorbit of a given $x_0 \in R$ is any (finite or infinite) sequence $x_0, x_{-1}, x_{-2}, \ldots$ such that $f(x_{-n}) = x_{-(n-1)}$ for all n for which x_{-n} is defined. The points $x_{-1}, x_{-2}, \ldots$ in any such sequence are called predecessors of x_0. A point c_0 is called critical if $f(c_0) = c_0$, i.e., a critical point of f is a fixed point of f. A periodic point x_0 of period $p > 1$ (p a positive integer) is a point for which the relations $f^p(x_0) = x_0$, $f^k(x_0) \neq x_0$, $1 \leq k < p$, hold. If x_0 is a periodic point of period p, its orbit is denoted by $(x_0, x_1, \ldots, x_{p-1})$. We shall denote the k-th iterate of x_0 under the function f^m by x_k^m, $k = 0, 1, \ldots$. Thus $x_k^m := (f^m)^k(x_0) = x_{mk}$, and, in particular, $x_0^m = x_k^0 = x_0$ for all nonnegative

integers k and m.
Definition. A periodic orbit $(x_0, x_1, \ldots, x_{n-1})$ of period n is called elementary if

$$x_{2\nu} < \cdots < x_4 < x_2 < x_0 < x_1 < x_3 < \cdots < x_{2k-1}$$

or

$$x_{2\nu} > \cdots > x_4 > x_2 > x_0 > x_1 > x_3 > \cdots > x_{2k-1},$$

where $\nu + 1 = k = \frac{n}{2}$ if n is even and $\nu = \frac{n-1}{2} = k$ if n is odd.
An infinite preorbit $(x_0, x_{-1}, x_{-2}, \ldots)$ is called elementary if the inequalities

$$x_{-2} < x_{-4} < \cdots < x_0 < \cdots < x_{-3} < x_{-1}$$

or

$$x_{-2} > x_{-4} > \cdots > x_0 > \cdots > x_{-3} > x_{-1}$$

hold.

Periodic orbits of period two or three are by definition elementary orbits.
We adopt the following notation : $P(k)$, $P(1)$, $E(k)$, $E(\infty)$, mean that f has a periodic orbit of period k, a fixed point, an elementary periodic orbit of period k, an infinite elementary preorbit, respectively. Similarly, $P^m(k)$, $E^m(k)$, $E^m(\infty)$, $m = 2, 3, \ldots$, mean that the m-th iterate f^m of f has a periodic orbit of period k, an elementary periodic orbit of period k, an infinite elementary preorbit, respectively.

3. Sarkovskii's Theorem and Theorem (SE).

Using the notation introduced in Section 2., Sarkovskii's theorem and its extension, Theorem (SE), read as follows.

Theorem (Sarkovskii). Let $f:\ R \to\ R$ be continuous. Then

$$\begin{array}{l} P(3) \Rightarrow P(5) \Rightarrow P(7) \Rightarrow \cdots \Rightarrow \\ P(2\cdot 3) \Rightarrow P(2\cdot 5) \Rightarrow P(2\cdot 7) \Rightarrow \cdots \Rightarrow \\ P(2^2\cdot 3) \Rightarrow P(2^2\cdot 5) \Rightarrow P(2^2\cdot 7) \Rightarrow \cdots \Rightarrow \\ \cdots \qquad\qquad\qquad\qquad \Rightarrow \cdots \Rightarrow \\ P(2^3) \Rightarrow P(2^2) \Rightarrow P(2) \Rightarrow P(1). \end{array}$$

Theorem (SE). Let $f:\ R \to R$ be continuous. Then

$$
\begin{array}{llllllllll}
P(3) \Rightarrow & P(5) \Rightarrow & \cdots \Rightarrow & E(\infty) \Rightarrow & \cdots \Rightarrow & E(8) \Rightarrow & E(6) \Rightarrow & E(4) \Rightarrow & P(2) \Rightarrow & P(1) \\
 & & & & & & & & \Uparrow & \\
P(2\cdot 3) \Rightarrow & P(2\cdot 5) \Rightarrow & \cdots \Rightarrow & E^2(\infty) \Rightarrow & \cdots \Rightarrow & E^2(8) \Rightarrow & E^2(6) \Rightarrow & E^2(4) \Rightarrow & P(2^2) \Rightarrow & P^2(1) \\
 & & & & & & & & \Uparrow & \\
P(2^2\cdot 3) \Rightarrow & P(2^2\cdot 5) \Rightarrow & \cdots \Rightarrow & E^{2^2}(\infty) \Rightarrow & \cdots \Rightarrow & E^{2^2}(8) \Rightarrow & E^{2^2}(6) \Rightarrow & E^{2^2}(4) \Rightarrow & P(2^3) \Rightarrow & P^{2^2}(1) \\
 & & & & & & & & \Uparrow & \\
P(2^3\cdot 3) \Rightarrow & P(2^3\cdot 5) \Rightarrow & \cdots \Rightarrow & E^{2^3}(\infty) \Rightarrow & \cdots \Rightarrow & E^{2^3}(8) \Rightarrow & E^{2^3}(6) \Rightarrow & E^{2^3}(4) \Rightarrow & P(2^4) \Rightarrow & P^{2^3}(1) \\
 & & & & & & & & \Uparrow & \\
 & & & & & \cdots & & & &
\end{array}
$$

4. Lemma F.

The following lemma states a condition that implies the existence of a periodic orbit. The formulation is a more specific version of a well-known "Folk-Lemma".

Lemma F. Let $f : R \to R$ be continuous and $L_1, L_2, \ldots, L_n$ compact intervals such that

$$f(L_i) \supset L_{i+1},\ i = 1, 2, \ldots, n-1$$

and

$$f(L_n) \supset L_1.$$

If $L_i \cap L_j,\ i \neq j$, is either empty or a singleton, then there is a point $x_0 \in L_1$ with $x_i \in L_{i+1},\ i = 1, 2, \ldots, n-1$, and $x_0 = x_n$. Such an x_0 has period n if n is odd and period n or $\frac{n}{2}$ if n is even, but the period $\frac{n}{2}$ is possible only if $x_i \in L_{i+1} \cap L_{(\frac{n}{2}+i+1)},\ i = 0, 1, 2, \ldots, \frac{n}{2} - 1$.

Proof. We recall first two basic facts :

(i) If f is continuous the inclusion $f(I) \supset J$ for compact intervals I and J implies the existence of a compact interval $I' \subset I$ such that $f(I') = J$.

(ii) If f is continuous the inclusion $f(K) \supset K$ for a compact interval K implies that f has a fixed point in K.

By (i) there are intervals $L_i' \in L_i$, $i = 1, 2, \ldots, n$ such that $f(L_n') = L_1$ and $f(L_i') = L_{i+1}'$, $i = 1, 2, \ldots, n-1$. Since $f^n(L_1') = f(L_n') = L_1 \supset L_1'$, we conclude by ($ii$) the existence of $x_0 \in L_1' \subset L_1$ with $x_i \in L_{i+1}$, $i = 1, 2, \ldots, n-1$, and $x_0 = x_n$. If n is odd, at least $\frac{1}{2}(n+1)$ of the L-intervals are mutually disjoint. Since any period of x_0, i.e., the number of distinct points in the orbit of x_0, is a divisor of n, x_0 has period n. If n is even and greater than 2, at least $\frac{n}{2}$ of the L-intervals are mutually disjoint. Consequently, x_0 has period n or $\frac{n}{2}$. If x_0 has period $\frac{n}{2}$, then $x_i = x_{\frac{n}{2}+i}$, $i = 0, 1, 2, \ldots, \frac{n}{2} - 1$. Since $x_i \in L_{i+1}$ and $x_i = x_{\frac{n}{2}+i} \in L_{\frac{n}{2}+i+1}$, we have $x_i \in L_{i+1} \cap L_{\frac{n}{2}+i+1}$, $i = 0, 1, 2, \ldots, \frac{n}{2} - 1$. This completes the proof.

5. The Hierarchy of the Elementary Orbits.

The following four propositions establish that the set of elementary orbits is totally ordered with respect to the relation "implies".

Proposition 1. $E(2n+1) \Rightarrow E(2n+3)$, $n = 1, 2, \ldots$.

Proof. Let $(x_0, x_1, \ldots, x_{2n})$ be an elementary $(2n+1)$-periodic orbit and assume without loss of generality that

$$x_{2n} < x_{2n-2} < \cdots < x_4 < x_2 < x_0 = x_{2n+1} < x_1 < x_3 < \cdots < x_{2n-3} < x_{2n-1}.$$

As a consequence of the intermediate value property of f we conclude the existence of a fixed point c_0 and predecessors c_{-1} and c_{-2} satisfying

$$x_{2n} < c_{-1} < x_{2n-2} < \cdots < x_2 < x_0 < c_0 < x_1 < \cdots < x_{2n-3} < c_{-2} < x_{2n-1} \tag{1}$$

if $n = 2, 3, \ldots$ and

$$x_2 < c_{-1} < x_0 < c_0 < c_{-2} < x_1 \tag{2}$$

if $n = 1$.
If $n \geq 3$, define the intervals $L_1 = [x_0, c_0]$, $L_2 = [c_0, x_1]$, $L_3 = [x_2, x_0]$, $\ldots$, $L_{2m} = [x_{2m-3}, x_{2m-1}]$, $L_{2m+1} = [x_{2m}, x_{2m-2}]$, $\ldots$, $L_{2n} = [x_{2n-3}, c_{-2}]$, $L_{2n+1} = [c_{-1}, x_{2n-2}]$, $L_{2n+2} = [c_{-2}, x_{2n-1}]$, and $L_{2n+3} = [x_{2n}, c_{-1}]$, $m = 2, 3, \ldots, n-1$.
If $n = 2$, define the intervals $L_1 = [x_0, c_0]$, $L_2 = [c_0, x_1]$, $L_3 = [x_2, x_0]$, $L_4 = [x_1, c_{-2}]$, $L_5 = [c_{-1}, x_2]$, $L_6 = [c_{-2}, x_3]$, and $L_7 = [x_4, c_{-1}]$.
If $n = 1$, define the intervals $L_1 = [x_0, c_0]$, $L_2 = [c_0, c_{-2}]$, $L_3 = [c_{-1}, x_0]$, $L_4 = [c_{-2}, x_1]$, and $L_5 = [x_2, c_{-1}]$.
Lemma F implies now the existence of an elementary orbit of period $2n + 3$ for $n = 1, 2, \ldots$. This completes the proof.
Remark. By choosing other L-intervals, it can be shown that there are two distinct elementary orbits of period $2n + 3$ for $n = 1, 2, \ldots$.

Proposition 2. $E(2n+1) \Rightarrow E(\infty)$, $n = 1, 2, \ldots$.

Proof. We may assume that the elementary $(2n+1)$-periodic orbit $E(2n+1)$ has the same orientation as the elementary $(2n+1)$-periodic orbit in the proof of Proposition 1, so that the inequalities (1) and (2) in Proposition 1. hold. Referring to inequality (1), the inequalities $f(x_{2n-2}) = x_{2n-1} > c_{-2}$ and $f(x_{2n-4}) = x_{2n-3} < c_{-2}$ imply that there is a $c_{-3} \in (x_{2n-2}, x_{2n-4})$. Referring to the inequality (2), the inequalities $f(x_0) = x_1 > c_{-2}$ and $f(c_0) = c_0 < c_{-2}$ imply a $c_{-3} \in (x_0, c_0)$. Hence, for $n \geq 1$, a fixed point $c_0 \in (x_0, x_1)$ and predecessors c_{-1}, c_{-2}, c_{-3} satisfying

$$c_{-1} < c_{-3} < c_0 < c_{-2}$$

always exist. We now find $c_{-4}, c_{-5}, \ldots$ successively. $f(c_{-2}) = c_{-1} < c_{-3} < c_0 = f(c_0)$ implies that there is a $c_{-4} \in (c_0, c_{-2})$, and from $c_0 = f(c_0) < c_{-4} < f(c_{-3}) = c_{-2}$ follows a $c_{-5} \in (c_{-3}, c_0)$. Repeating these arguments, we conclude that $(c_0, c_{-1}, c_{-2}, \ldots)$ is an elementary infinite preorbit. Hence, $E(2n+1) \Rightarrow E(\infty)$ and the proof is complete.

Proposition 3. $E(\infty) \Rightarrow E(2m)$, $m = 1, 2, \ldots$.

Proof. Let $(x_0, x_{-1}, x_{-2}, \ldots)$ be an infinite preorbit satisfying

$$x_{-2} < x_{-4} < x_{-6} < \cdots < x_0 < \cdots < x_{-5} < x_{-3} < x_{-1}$$

Define $L_{2m} = [x_{-3}, x_{-1}]$, $L_{2m-1} = [x_{-2}, x_{-4}]$, $L_{2m-2} = [x_{-5}, x_{-3}]$, $L_{2m-3} = [x_{-4}, x_{-6}], \ldots$, $L_2 = [x_{-(2m+1)}, x_{-(2m-1)}]$, and $L_1 = [x_{-2m}, x_{-(2m+2)}]$. Lemma F implies now the existence of an elementary orbit of period $2m$. This completes the proof.

Proposition 4. $E(2m+2) \Rightarrow E(2m)$, $m = 1, 2, \ldots$.

Proof. If $(x_0, x_1, \ldots, x_{2m+1})$ is the elementary orbit of period $2m+2$, we may assume that

$$x_{2m} < \cdots < x_2 < x_0 < x_1 < \cdots < x_{2m+1}.$$

If we let $L_1 = [x_2, x_0]$, $L_2 = [x_1, x_3]$, $L_3 = [x_4, x_2]$, $L_4 = [x_3, x_5], \ldots, L_{2m-1} = [x_{2m}, x_{2m-2}]$, and $L_{2m} = [x_{2m-1}, x_{2m+1}]$, Lemma F guarantees an elementary orbit of period $2m$ and the proof is complete.
Remark. It can be shown that there exist two distinct elementary orbits of period $2m$ for $m = 3, 4, \ldots$.
Summarizing the Propositions 1., 2., 3., and 4., we obtain the following Theorem 1..

Theorem 1. (The total ordering of elementary orbits).

$$E(3) \Rightarrow E(5) \Rightarrow \cdots \Rightarrow E(\infty) \Rightarrow \cdots \Rightarrow E(8) \Rightarrow E(6) \Rightarrow E(4) \Rightarrow E(2) \Rightarrow E(1).$$

Remark. We notice that the total ordering of the periodic elementary orbits is different from Sarkovskii's ordering.

6. The Relationship of $P(n)$ to $E(n)$.

Among the points of a periodic orbit P we denote by P^+ the set of those points x in P with $f(x) > x$ and by P^- the set of points in P with $f(x) < x$. P^+ and P^- are nonempty for any periodic orbit P and we let $m = m^+ = min\ P^+$, $M^+ = max\ P^+$, $m^- = min\ P^-$, $M = M^- = max\ P^-$. If we define the "spans" of P, P^+, and P^- by span $P = [m, M]$, span $P^+ = [m^+, M^+]$, span $P^- = [m^-, M^-]$, then for any P one of the following cases occurs:

(a) span $P^+ \cap$ span $P^- \neq \emptyset$

or

(b) span $P^+ \cap$ span $P^- = \emptyset$.

If (a) occurs, then span P contains three distinct "forced" fixed points, one each in the intervals (m, m^-), (m^-, M^+), and (M^+, M). This situation ensures always the existence of a three-periodic orbit as follows from the slightly more general Theorem 2. below. If case (b) occurs, only one "forced" fixed point in the span of P is assured and, in general, no conclusion for the existence of an elementary periodic orbit other than $E(2)$ or $E(4)$ is possible. However, if the periodic orbit P is such that $P^+ \cap f(P^+) \neq \emptyset$ (or $P^- \cap f(P^-) \neq \emptyset$), then an odd elementary periodic orbit is necessarily present. This will be the case, for example, if the period of P is odd. This fundamental situation is dealt with in Theorem 3.. It is this theorem and its corollaries which provides one of the crucial steps from the hierarchy of elementary orbits to Theorem (SE) and, hence, to Sarkovskii's theorem.

Theorem 2. If P is a periodic orbit of f such that its span contains two fixed points separated by at least one point of the orbit, then f has a three-periodic orbit.

Proof. Let $[m, M]$ be the span of P and c_0, d_0 fixed points of f such that

$$m < c_0 < a < d_0 < M,$$

where a is a point of the orbit. We may assume that the interval (c_0, d_0) contains only points x of the orbit with $f(x) > x$ and the interval (d_0, M) contains only points y of the orbit with $f(y) < y$. Let a be that point of the orbit in (c_0, d_0) with the largest image $f(a)$. Then there is a point b of the orbit in the interval $(d_0, f(a)]$ such that $f(b) < c_0$ (otherwise, the orbit of a would not contain m). Then $L_1 = [c_0, a]$, $L_2 = [a, d_0]$, and $L_3 = [d_0, b]$ define a three-periodic orbit of f by Lemma F. This completes the proof.
Remark. It is worthwhile to analyze the hypothesis of this theorem somewhat further and inquire about the nature of the fixed point c_0. Then it follows that c_0 gives rise to the existence of an infinite preorbit called an infinite loop, a much stronger conclusion (see[6] for the notion of infinite loop and its equivalence with the notion of turbulence [6]).

Theorem 3. Let P be a periodic orbit of period $n \geq 3$. If the spans of P^+ and P^- are disjoint and $f(P^+) \cap P^+ \neq \emptyset$ (or $f(P^-) \cap P^- \neq \emptyset$), then f has an odd elementary orbit of period $2k + 1 \leq n$.

Proof. We order the points in P^+ in descending order, the points in P^- in ascending order as follows:

$$x^p \leq x^{i+1} < x^i < \cdots < x^1 < c_0 < y^1 < y^2 < \cdots < y^q,$$

where c_0 is a critical point, $p+q = n$, and x^{i+1} denotes the largest point in P^+ with the property $f(x^{i+1}) \in P^+$. Thus $i \geq 1$. We let $z^1 = x^1$, $L_1 = [z^1, c_0]$ and select successively unique points $z^2 \in P^-$, $z^3 \in P^+$ with $z^3 < z^1$ and let $L_2 = [c_0, z^2]$, $L_3 = [z^3, z^1]$ as follows: $z^2 = max\ f(L_1) \cap P^-$, $z^3 = min\ f(L_2) \cap P^+$. We note that z^2 is well-defined as $z^1 = x^1$ and $f(x^1) \in P^-$ ensures $f(L_1) \cap P^- \neq \emptyset$ as well as $y^1 \leq z^2 \leq y^q$. Furthermore, z^3 is well-defined with $z^3 < z^1$ since $y^1 \in L_2$ ensures $f(L_2) \cap P^+ \neq \emptyset$, and so $z^3 \leq z^1 \leq x^1$, but $z^3 = z^1$ forces the orbit of x^1, that is P, to be contained in $L_1 \cup L_2$, whereas $x^{i+1} \notin L_1 \cup L_2$, a contradiction. Hence $z^3 < z^1$. We observe next that $f(L_1) \supset L_2$, $f(L_2) \supset L_3$. If further $z^3 \leq x^{i+1}$, we also have $f(L_3) \supset L_1$ and conclude a three-periodic orbit by lemma F (this happens, for example, if $z^2 = y^q$). If however, $z^3 > x^{i+1}$, we select successively unique points $z^4 \in P^-$ with $z^2 < z^4$ and $z^5 \in P^+$ with $z^5 < z^3$ as follows: $z^4 = max\ f(L_3) \cap P^-$, $z^5 = min\ f(L_4) \cap P^+$. Letting $L_4 = [z^2, z^4]$, $L_5 = [z^5, z^3]$, we have $f(L_1) \supset L_2$, $f(L_2) \supset L_3$, $f(L_3) \supset L_4$, and $f(L_4) \supset L_5$. If then further $z^5 \leq x^{i+1}$, we also have $f(L_5) \supset L_1$, and hence f has an elementary periodic orbit of period five by lemma F. In the case $z^5 > x^{i+1}$, we continue the selection process as shown for the first two stages. The process must end with the selection of an odd number of points $z^1, z^2, \cdots, z^{2k+1}$ of the orbit satisfying

$$x^p \leq z^{2k+1} \leq x^{i+1} < z^{2k-1} < \cdots < z^3 < z^1 = x_1 < c_0 < y^1 \leq z^2 < z^4 < \cdots < z^{2k} \leq y^q,$$

in which case lemma F ensures that f has an elementary periodic orbit of period $2k+1 \leq n$. This completes the proof.

Theorem 3. implies the following theorems as corollaries.

Theorem 4. $P(2n+1) \Rightarrow E(2n+1)$.

Theorem 5. If f has an odd period that is not elementary, $P(2n+1) \neq E(2n+1)$, then $P(2n+1) \Rightarrow E(2k+1)$ for some k with $1 \leq k < n$.

As a consequence of this theorem we obtain the theorem that is known as Stefan's theorem.

Theorem 6 (Stefan). If f has a periodic orbit of period $2n+1$ but no periodic orbit of period $2k+1$, $k < n$, then $P(2n+1) = E(2n+1)$.

Remark. Periodic orbits for which the hypotheses of theorems 2. and 3. do not hold are even orbits for which $P^+ = f^2(P^+)$ holds and include all elementary even periodic orbits.

7. The Propositions $P^2(n) \Leftrightarrow P(2n)$ and $P(4) \Rightarrow E(4)$.

Proposition 5. $P^2(n) \Leftrightarrow P(2n)$, $n \geq 2$.

Proof. $P(2n) \Rightarrow P^2(n)$: Let $(x_0, x_1, \ldots, x_{2n-1})$, $x_0 = x_{2n}$, be any $2n$-periodic orbit of f. Then, setting $x_0^2 = x_0$, we note that $(x_0^2, x_1^2, \ldots, x_{n-1}^2)$ is an n-periodic orbit of f^2. Hence, $P(2n) \Rightarrow P^2(n)$.
$P^2(n) \Rightarrow P(2n)$: Let $(x_0^2, x_1^2, \ldots, x_{n-1}^2)$, $x_0^2 = x_n^2$, be an n-periodic orbit of f^2. Then, setting $x_0 = x_0^2$, we obtain the periodic orbit $(x_0, x_1, x_2, \ldots, x_{2n-1})$, $x_0 = x_{2n}$, which has a period n'. n' is either $2n$ or a divisor of $2n$. If $n' \neq 2n$, then n' must be an odd number $n' \geq 3$ (because an even n' forces $(x_0^2, x_1^2, \ldots, x_{n-1}^2)$, $x_0^2 = x_n^2$, to have period less than n). Thus f has a periodic orbit of period $2n$ or a periodic orbit of period $n' = 2k+1 \geq 3$. In the latter case Theorems 4. and 1. guarantee that f has a $2n$-periodic orbit since $P(n') = P(2k+1) \Rightarrow E(2k+1) \Rightarrow E(2m)$ for all $k \geq 1$ and $m = 1, 2, \ldots$. Hence, $P^2(n) \Rightarrow P(2n)$. This completes the proof.

Corollary. $P^{2^m}(n) \Leftrightarrow P(2^m n)$, $m = 1, 2, \ldots$, $n = 2, 3, \ldots$.

Proposition 6. $P(4) \Rightarrow E(4)$.

Proof. Of the six types of four-periodic orbits two are elementary. The

remaining four

$$(i) \quad x_4 = x_0 < x_1 < x_2 < x_3$$

$$(ii) \quad x_4 = x_0 < x_3 < x_2 < x_1$$

$$(iii) \quad x_4 = x_0 < x_1 < x_3 < x_2$$

$$(iv) \quad x_4 = x_0 < x_3 < x_1 < x_2$$

define each three natural intervals that satisfy the conditions of Lemma F, and, therefore, ensure the existence of a 3-periodic orbit, which, in turn, implies an $E(4)$ by Theorem 1.. These intervals are defined by

$$L_1 = [x_0, x_1], \quad L_2 = [x_1, x_2], \quad L_3 = [x_2, x_3] \quad \text{in case (i)},$$

$$L_1 = [x_2, x_1], \quad L_2 = [x_3, x_2], \quad L_3 = [x_0, x_3] \quad \text{in case (ii)},$$

$$L_1 = [x_0, x_1], \quad L_2 = [x_1, x_3], \quad L_3 = [x_3, x_2] \quad \text{in case (iii)}.$$

In the last case, (iv), we notice first fixed points $c_0 \in [x_3, x_1]$ and $d_0 \in [x_1, x_2]$ and then let $L_1 = [c_0, x_1]$, $L_2 = [x_1, d_0]$, and $L_3 = [d_0, x_2]$. This completes the proof.
Since $E^2(2) \Rightarrow P(4) \Rightarrow E(4) \Rightarrow E(2)$, we obtain the following important corollary.

Corollary. $E^2(2) \Rightarrow E(2)$.

8. Principal Results.

If we consider the total ordering of the elementary orbits for the functions $f, f^2, \cdots, f^{2^n}, f^{2^{n+1}}, \cdots$, then the implications $E(6) \Rightarrow P(6) \Rightarrow P^2(3) \Rightarrow E^2(3)$ and $E^2(2) \Rightarrow E(2)$ provide the linkage shown in the following result.

Theorem 7. Let $f : R \to R$ be continuous. Then

$$
\begin{array}{ccccccccccccccccc}
E(3) & \Rightarrow & E(5) & \Rightarrow & \cdots & \Rightarrow & E(\infty) & \Rightarrow & \cdots & \Rightarrow & E(6) & \Rightarrow & E(4) & \Rightarrow & E(2) & \Rightarrow & E(1) \\
\Downarrow & & & & & & & & & & & & & & \Uparrow & & \\
E^2(3) & \Rightarrow & E^2(5) & \Rightarrow & \cdots & \Rightarrow & E^2(\infty) & \Rightarrow & \cdots & \Rightarrow & E^2(6) & \Rightarrow & E^2(4) & \Rightarrow & E^2(2) & \Rightarrow & E^2(1) \\
\Downarrow & & & & & & & & & & & & & & \Uparrow & & \\
 & & & & & & & & \cdots & & & & & & & & \\
\Downarrow & & & & & & & & & & & & & & \Uparrow & & \\
E^{2^n}(3) & \Rightarrow & E^{2^n}(5) & \Rightarrow & \cdots & \Rightarrow & E^{2^n}(\infty) & \Rightarrow & \cdots & \Rightarrow & E^{2^n}(6) & \Rightarrow & E^{2^n}(4) & \Rightarrow & E^{2^n}(2) & \Rightarrow & E^{2^n}(1) \\
\Downarrow & & & & & & & & & & & & & & \Uparrow & & \\
E^{2^{n+1}}(3) & \Rightarrow & E^{2^{n+1}}(5) & \Rightarrow & \cdots & \Rightarrow & E^{2^{n+1}}(\infty) & \Rightarrow & \cdots & \Rightarrow & E^{2^{n+1}}(6) & \Rightarrow & E^{2^{n+1}}(4) & \Rightarrow & E^{2^{n+1}}(2) & \Rightarrow & E^{2^{n+1}}(1) \\
\Downarrow & & & & & & & & & & & & & & \Uparrow & & \\
 & & & & & & & & \cdots & & & & & & & & .
\end{array}
$$

Since the implications $P(2n+1) \Longleftrightarrow E(2n+1)$, $E(2) \Longleftrightarrow P(2)$, and $P^2(n) \Longleftrightarrow P(2n)$, ensure the implications $E^{2^n}(2m+1) \Longleftrightarrow P((2m+1)2^n)$ and $E^{2^n}(2) \Longleftrightarrow P(2^{n+1})$, we obtain, as stated in Section 3., the following extension of Sarkovskii's theorem.

Theorem (SE). Let $f: R \to R$ be continuous. Then

$$
\begin{array}{ccccccccccccccccccc}
P(3) & \Rightarrow & P(5) & \Rightarrow & \cdots & \Rightarrow & E(\infty) & \Rightarrow & \cdots & \Rightarrow & E(8) & \Rightarrow & E(6) & \Rightarrow & E(4) & \Rightarrow & P(2) & \Rightarrow & P(1) \\
\Downarrow & & & & & & & & & & & & & & & & \Uparrow & & \\
P(2\cdot 3) & \Rightarrow & P(2\cdot 5) & \Rightarrow & \cdots & \Rightarrow & E^2(\infty) & \Rightarrow & \cdots & \Rightarrow & E^2(8) & \Rightarrow & E^2(6) & \Rightarrow & E^2(4) & \Rightarrow & P(2^2) & \Rightarrow & P^2(1) \\
\Downarrow & & & & & & & & & & & & & & & & \Uparrow & & \\
P(2^2\cdot 3) & \Rightarrow & P(2^2\cdot 5) & \Rightarrow & \cdots & \Rightarrow & E^{2^2}(\infty) & \Rightarrow & \cdots & \Rightarrow & E^{2^2}(8) & \Rightarrow & E^{2^2}(6) & \Rightarrow & E^{2^2}(4) & \Rightarrow & P(2^3) & \Rightarrow & P^{2^2}(1) \\
\Downarrow & & & & & & & & & & & & & & & & \Uparrow & & \\
P(2^3\cdot 3) & \Rightarrow & P(2^3\cdot 5) & \Rightarrow & \cdots & \Rightarrow & E^{2^3}(\infty) & \Rightarrow & \cdots & \Rightarrow & E^{2^3}(8) & \Rightarrow & E^{2^3}(6) & \Rightarrow & E^{2^3}(4) & \Rightarrow & P(2^4) & \Rightarrow & P^{2^3}(1) \\
\Downarrow & & & & & & & & & & & & & & & & \Uparrow & & \\
 & & & & & & & & \cdots & & & & & & & & & & .
\end{array}
$$

Sarkovskii's theorem follows now as a consequence of Theorem (SE) by omitting all E-implications.

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